Harmonic Maps and Dirac-harmonic Maps from Degenerating Surfaces

Summary of Dissertation

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In this thesis, we study the compactness of harmonic maps and Dirac-harmonic maps from degenerating surfaces.

In the first part, we consider a sequence of harmonic maps

$$\phi_n : (M_n, c_n) \to (N, g)$$

with uniformly bounded energy $E(\phi_n, M_n) = \int_{M_n} |d\phi_n|^2 \leq \Lambda$, where (M_n, c_n) is a sequence of closed Riemann surfaces of genus $p > 1$ with complex structures c_n and (N, g) is a compact Riemannian manifold. The uniform energy bound implies weak subconvergence of ϕ_n. In general, strong convergence fails because of possible energy concentration. When the complex structures vary in a compact region of the moduli space $\mathcal{M}_p$, the concentrated energy can be captured by finitely many bubbles, i.e. harmonic maps from S^2. This yields an energy identity, which means the necks connecting the weak limit and the bubbles contain no energy – a well known result by Sacks-Uhlenbeck, Jost and Parker. When the complex structures degenerate, the energy identity remains hold if ϕ_n are conformal or ϕ_n is an energy minimizing sequence in the same homotopy class. However, in general, energy may get lost from some necks and hence the energy identity fails to hold.

In this thesis, we develop an approach to study the degenerating case and establish a generalized energy identity by adding a correction term to the classical one. More precisely, we assume that (M_n, c_n) degenerates to a punctured Riemann surface (M, c) by collapsing a simple closed geodesic γ_n. Let h_n be the hyperbolic metric compatible with c_n, denote the h_n-length of γ_n by l_n, and let P_n be the standard cylindrical collar about γ_n. Our main observation is that one can associate to (ϕ_n, M_n) a sequence of quantities $\alpha_n := \alpha(\phi_n, P_n) \in \mathbb{C}$ defined by

$$\alpha(\phi, P) := \int_{\{t\} \times S^1} T(\phi) d\theta,$$

for a harmonic map ϕ on a cylinder $P = [t_1, t_2] \times S^1$ with the Hopf differential $T(\phi)dz^2$, where $z = t + i\theta$. Here $\alpha(\phi, P)$ does not depend on $t \in [t_1, t_2]$. Then, we have

Theorem 1. *There exist finitely many harmonic maps: $\phi : (\overline{M}, \overline{c}) \to N$, where $(\overline{M}, \overline{c})$ is the normalization of (M, c); $\sigma^i : S^2 \to N, i = 1, 2, ..., I$, such that, after selection of a subsequence, the following holds:*

$$\lim_{n\to\infty} E(\phi_n) \;=\; E(\phi) + \sum_{i=1}^{I} E(\sigma^i) + \lim_{n\to\infty} |\mathrm{Re}\alpha_n| \cdot \frac{\pi^2}{l_n}$$

During the blow-up process, some necks connecting ϕ and the bubbles $\{\sigma^i\}$ appear. Theorem 1 shows that the limit of the total energy of these necks is $\lim_{n\to\infty} |\mathrm{Re}\alpha_n| \cdot \frac{\pi^2}{l_n}$. Moreover, we can estimate the total lengths of the necks in terms of (α_n, l_n) and then obtain the following result

Theorem 2. *(ϕ_n, M_n) subconverges in C^0 modulo bubbles, i.e., in the limit, all necks converge to points in the target N, if and only if* $\liminf_{n\to\infty} \sqrt{|\mathrm{Re}\alpha_n|} \cdot \frac{\pi^2}{l_n} = 0$.

In the second part of the thesis, we apply similar ideas and methods to the study of the compactness of Dirac-harmonic maps from degenerating spin surfaces.

Motivated by the supersymmetric nonlinear sigma model from quantum field theory, Dirac-harmonic maps are defined as critical points of the action functional

$$L(\phi,\psi) = \int |d\phi|^2 + \int \langle \not{D}\psi, \psi\rangle,$$

where ϕ is a map from a Riemann surface (M,h) with a spin structure $\mathfrak{S}$ to a Riemannian manifold (N,g) and ψ is a section of the bundle $\Sigma M \otimes \phi^*TN \to M$ where ΣM is the usual spinor bundle on M. The functional L is conformally invariant. Consequently, the analysis of Dirac-harmonic maps has the typical features of conformally invariant variational problems.

To study the compactness of Dirac-harmonic maps, we consider a sequence of Dirac-harmonic maps $(\phi_n, \psi_n) : (M_n, h_n, \mathfrak{S}_n) \to (N,g)$ with uniformly bounded energy

$$E(\phi_n, \psi_n, M_n) = \int_{M_n} (|d\phi_n|^2 + |\psi_n|^4) \leq \Lambda,$$

where $(M_n, h_n, \mathfrak{S}_n)$ is a sequence of closed Riemann spin surfaces of genus $p > 1$. Following Sacks-Uhlenbeck's blow-up analysis, one can obtain bubbling solutions. When the domain is fixed, the corresponding energy identities of (ϕ_n, ψ_n) were established by Chen-Jost-Li-Wang and Zhao.

In this thesis, the degenerating case of Dirac-harmonic maps is studied. We assume that $(M_n, h_n, \mathfrak{S}_n)$ degenerates to a punctured spin surface $(M, h, \mathfrak{S})$ by collapsing a simple closed geodesic γ_n of length l_n. As in the case of harmonic maps, we associate to (ϕ_n, ψ_n, M_n) a sequence of quantities $\alpha_n := \alpha(\phi_n, \psi_n, P_n) \in \mathbb{C}$ defined by

$$\alpha(\phi,\psi,P) := \int_{\{t\}\times S^1} T(\phi,\psi) d\theta.$$

Here $T(\phi,\psi)dz^2$ is the generalized Hopf differential of (ϕ,ψ) on P. Then, we have

Theorem 3. *Assume that all punctures of the limit surface are of Neveu-Schwarz type and hence $\mathfrak{S}$ extends to some spin structure $\overline{\mathfrak{S}}$ on the normalization $\overline{M}$ of M. Then there exist finitely many Dirac-harmonic maps: $(\phi,\psi) : (\overline{M}, \overline{c}, \overline{\mathfrak{S}}) \to N$; $(\sigma^i, \xi^i) : S^2 \to N, i = 1, 2, ..., I$, such that after selection of a subsequence, the following hold:*

$$\begin{aligned} \lim_{n\to\infty} E(\phi_n) &= E(\phi) + \sum_{i=1}^{I} E(\sigma^i) + \lim_{n\to\infty} |\mathrm{Re}\alpha_n| \cdot \frac{2\pi^2}{l_n} \\ \lim_{n\to\infty} E(\psi_n) &= E(\psi) + \sum_{i=1}^{I} E(\xi^i). \end{aligned}$$