Zusammenfassung der Ergebnisse zur Dissertation

Weak-noise-induced phenomena in a slow-fast dynamical system

von Marius E. Yamakou

Max-Planck-Institut für Mathematik in den Naturwissenschaften, Leipzig

E-mail: yamakou@mis.mpg.de

This thesis considers a stochastic dynamical system that has multiple time scales. The focus lies on the planar slow-fast stochastic differential equation (a singularly perturbed stochastic system) modeling the spiking activity of an isolated FitzHugh-Nagumo neural oscillator. We analyze the model equation on the slow time scale τ (Eq.(1)) and on the fast time scale t (Eq.(2)).

$$\begin{cases} dv_\tau &= \varepsilon^{-1} f(v_\tau, w_\tau)d\tau + \frac{\sigma}{\sqrt{\varepsilon}} dW_\tau, \\ dw_\tau &= g(v_\tau, w_\tau)d\tau, \end{cases} \tag{1}$$

$$\begin{cases} dv_t &= f(v_t, w_t)dt + \sigma dW_t, \\ dw_t &= \varepsilon g(v_t, w_t)dt, \end{cases} \tag{2}$$

where the deterministic velocity vector field is given precisely by

$$\begin{cases} f(v,w) &= -av + (a+1)v^2 + ev^3 + fw, \\ g(v,w) &= bv + d - cw. \end{cases} \tag{3}$$

The phase-space consists of $(v, w) \in \mathbb{R}^2$ representing the activity of the fast action potential variable v and slow recovery current variable w.

The deterministic parameter-space consists of $\big(a, b, c, d, e, f, \varepsilon\big) \in \mathbb{R}^7$. Parameter a is often confined to the range $0 < a \leq 1$, but in this thesis, $a < 0$ will also be considered. $c > 0$ is a codimension-one Hopf bifurcation parameter. $0 < \varepsilon := \tau/t \ll 1$ is a singular perturbation parameter, the time-scale separation ratio between τ and t. This is a very important parameter and plays a big role in the analysis and results.

dW_t is standard Gaussian white noise, the formal derivative of Brownian motion with mean zero and unit variance, and $\sigma \geq 0$ is the amplitude of this random perturbation.

Associated to zero-noise equation of Eq.(1) (or Eq.(2)), the following three important sets are defined:

- critical manifold: $\mathcal{M}_0 := \big\{(v, w) \in \mathbb{R}^2 : f(v, w) = 0\big\}$,

- fixed points: $(v_e, w_e) := \big\{(v, w) \in \mathbb{R}^2 : f(v, w) = g(v, w) = 0\big\}$,

- non-hyperbolic singularities of $\mathcal{M}_0$: $(v_f, w_f) := \big\{(v, w) \in \mathbb{R}^2 : D_v f(v, w) = 0\big\}$.

This thesis analyzes in detail two different weak-noise-induced phenomena in model of Eq.(1) and provides the mathematical relationship between them [1, 2, 3].

Weak-noise-induced transitions with inhibition and modulation of oscillations (ISR) [1]

In the first part of this thesis, we analyze the effect of weak-noise-induced transitions on the dynamics of Eq.(1) in a bistable state consisting of a stable fixed point and a stable unforced limit cycle (spiking activity of the neuron). In the parametric zone of bi-stability, weak σ may strongly inhibit the neuron's spiking activity. Surprisingly, increasing σ leads to a minimum in the spiking rate N, after which N starts to increase monotonically with increase in σ. We investigate this inhibition and modulation of oscillations by looking at the variation of the mean number of spikes per unit time with σ. Bifurcation and slow-fast analysis give conditions on the parameter space for the establishment of a bi-stability regime. We show that this phenomenon always occurs when the initial conditions lie in the basin of attraction of the stable limit cycle. For initial conditions in the basin of attraction of the stable fixed point, the phenomenon however disappears, unless ε is bounded within some interval. We provide a theoretical explanation of ISR in terms of the stochastic sensitivity functions of the attractors and their minimum Mahalanobis distances from the separatrix isolating the basins of attraction.

Coherent oscillations induced by weak noise (SISR) [2]

In the second part of this thesis, we analyze the emergence of noise-induced coherent oscillations in Eq.(1). With our choice of the parameters of the deterministic equation, there exists a unique fixed point, which is located to the left of the left non-hyperbolic singularity of $\mathcal{M}_0$. We bound c such that a limit cycle solution cannot emerge through a singular Hopf bifurcation. After imposing a strong time-scale separation ($\varepsilon \to 0$) between the slow and fast variables, σ in a weak limit ($\sigma \to 0$) may induce a limit cycle behavior (coherent spike train). Due to our parameter settings, such a limit cycle behavior is absent in the dynamics of the noise-free model equation. We investigate the mechanism behind this phenomenon by using multiple-time perturbation techniques and by analyzing the escape mechanism of the random trajectories from the stable manifolds of the model equation. We determine the interval in which σ has to lie in order to induce a limit cycle in the limit as $\varepsilon \to 0$. We find another very small interval for σ in which the neuron generates only a Poisson sequence of spikes in the limit as $\varepsilon \to 0$. We also find that decreasing the value of ε does not significantly increase the coefficient of variation of the oscillations, but rather increases the interval of σ for which coherence occurs. Moreover, the phenomenon is robust under parameter tuning. Numerical simulations exhibit the results predicted by the theoretical analysis.

A simple parameter can switch between ISR and SISR [3]

The result in this part deduces the precise parametric perturbations that will allow the neuron to switch from SISR to ISR (and vice versa) in the same weak noise limit. These parametric variations boil down to just (carefully) changing the relative position of the unique fixed point with respect to the left non-hyperbolic singularity. This finding could explain why real biological neurons having similar physiological features and synaptic inputs may encode very different information [**4**].

References

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