Noncommutative Iwasawa Main Conjectures for Varieties over Finite Fields

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We state and prove an analogue for varieties over finite fields of T. Fukaya's and K. Kato's version of the noncommutative Iwasawa main conjecture. Moreover, we explain how this statement can be reinterpreted in terms of Waldhausen K*-theory.*

Let $\mathbb{F}$ be the field with $q = p^\mu$ elements and let $\mathfrak{A}$ denote the class of compact, semilocal rings Λ whose Jacobson radical is finitely generated and such that p is a unit in Λ. Examples are $\mathbb{Z}_\ell$ for $\ell \neq p$ and the noncommutative generalisations of the Iwasawa algebra. Write $\mathbf{P}(\Lambda)$ for the category of perfect complexes of left Λ-modules.

In this thesis, we construct a Waldhausen category $\mathbf{P}(X, \Lambda)$ of perfect complexes of Λ-adic sheafs for each variety X over $\mathbb{F}$ and for each Λ in $\mathfrak{A}$. In the classical case $\Lambda = \mathbb{Z}_\ell$, the category $\mathbf{P}(X, \mathbb{Z}_\ell)$ is closely related to P. Deligne's derived category of ℓ-adic sheaves.

Furthermore, we construct Waldhausen exact functors that correspond to the usual derived functors featuring in étale cohomology theory. For example, we show that there exists a functor

$$\mathfrak{R}\Gamma_c(X, -)\colon \mathbf{P}(X, \Lambda) \to \mathbf{P}(\Lambda)$$

whose cohomology modules agree with cohomology with proper support.

If Λ is commutative and if $\mathcal{F}^\bullet$ is a complex in $\mathbf{P}(X, \Lambda)$, the classical construction yields the L-function $L(\mathcal{F}^\bullet, T)$ of $\mathcal{F}^\bullet$ as a unit in $\Lambda[[T]]$:

$$L(\mathcal{F}^\bullet, T) \in \Lambda[[T]]^\times = \mathrm{K}_1(\Lambda[[T]]).$$

Moreover, if $S \subset \Lambda$ is the set of nonzero divisors and if the localisation $S^{-1}\mathfrak{R}\Gamma(X, \mathcal{F}^\bullet)$ for $\mathcal{F}^\bullet \in \mathbf{P}(X, \Lambda)$ is acyclic over $S^{-1}\Lambda$, one can evaluate the L-function at 1:

$$L(\mathcal{F}^\bullet, 1) \in S^{-1}\Lambda^\times = \mathrm{K}_1(S^{-1}\Lambda).$$

The thesis contains an explanation how this generalises to noncommutative rings Λ in $\mathfrak{A}$ and to arbitrary left denominator sets $S \subset \Lambda$.

Let $\mathbb{K}(\mathbf{W})$ be the topological space whose homotopy groups are the K-groups of the Waldhausen category $\mathbf{W}$.

Conjecture 1. *There exists a canonical system of homotopies*

$$\mathbb{K}\mathbf{P}(X, \Lambda) \underset{0}{\overset{\mathbb{K}\mathfrak{R}\Gamma_c(X,-)}{\genfrac{}{}{0pt}{}{\longrightarrow}{\Downarrow z_{X,\Lambda}\ \longrightarrow}}} \mathbb{K}\mathbf{P}(\Lambda)$$

for each ring Λ in $\mathfrak{A}$ and each variety X over $\mathbb{F}$. This system satisfies the following properties:

(1) *The $z_{X,\Lambda}$ are compatible under changes of Λ.*
(2) *The $z_{X,\Lambda}$ are compatible under morphisms $f\colon X \to Y$.*

(3) *If S is a left denominator set in Λ such that $S^{-1}\mathfrak{R}\Gamma_c(X,\mathcal{F}^\bullet)$ is acyclic, then the composition of the path associated to $0 \xrightarrow{\sim} S^{-1}\mathfrak{R}\Gamma_c(X,\mathcal{F}^\bullet)$ with the path $z_{X,\Lambda}(\mathcal{F}^\bullet)$ defines a loop class in $\mathrm{K}_1(S^{-1}\Lambda)$ that coincides with $L(\mathcal{F}^\bullet,1)$.*

Let $P_1(\mathbb{K}(\mathbf{W}))$ denote the 1-type of $\mathbb{K}(\mathbf{W})$, i. e

$$\pi_i(P_1(\mathbb{K}(\mathbf{W}))) = \begin{cases} \pi_i(\mathbb{K}(\mathbf{W})) & \text{for } i = 0,1; \\ 0 & \text{for } i > 1. \end{cases}$$

We prove:

Theorem 2 (Analogue of Fukaya's and Kato's conjecture)**.** *Conjecture 1 is true for the 1-type of the* K*-theory spaces. In this situation, the system of homotopies $z_{X,\Lambda}$ is uniquely determined by the properties (1)–(3).*

Theorem 3. *There exists a homotopy*

$$\mathbb{K}\mathbf{P}(X,\Lambda) \underset{0}{\overset{\mathbb{K}\mathfrak{R}\Gamma_c(X,-)}{\Downarrow}} \mathbb{K}\mathbf{P}(\Lambda)$$

for each ring Λ in $\mathfrak{A}$ and each variety X.

It remains to prove that there exists a choice of the homotopies in Theorem 3 that satisfies the compatibility conditions (1)–(3).

From Theorem 2, we deduce the following formula for the noncommutative L-values: For any left denominator set $S \subset \Lambda$ we may consider the exact sequence of relative K-theory

$$\cdots \to \mathrm{K}_1(\Lambda) \to \mathrm{K}_1(S^{-1}\Lambda) \xrightarrow{d} \mathrm{K}_0(\Lambda, S^{-1}\Lambda) \to \cdots$$

Theorem 4. *Assume that $S^{-1}\mathfrak{R}\Gamma_c(X,\mathcal{F}^\bullet)$ is acyclic. Then*

$$dL(\mathcal{F}^\bullet,1) = [\mathfrak{R}\Gamma_c(X,\mathcal{F}^\bullet)]^{-1} \in \mathrm{K}_0(\Lambda, S^{-1}\Lambda).$$

For $\Lambda = \mathbb{Z}_\ell$ and $S^{-1}\Lambda = \mathbb{Q}_\ell$, the above sequence can be identified with

$$\mathbb{Z}_\ell^\times \to \mathbb{Q}_\ell^\times \xrightarrow{x \mapsto |x|_\ell} \ell^{\mathbb{Z}},$$

where $|x|_\ell$ denotes the ℓ-adic absolute value. Write $\mathbb{Z}_\ell(n)_X$ for the object in $\mathbf{P}(X,\mathbb{Z}_\ell)$ corresponding to the n-th Tate twist of the constant sheaf $\mathbb{Z}_\ell$. Then

$$L(\mathbb{Z}_\ell(n)_X, T) = Z(X, q^{-n}T),$$

where $Z(X,t)$ is the Zeta function of X.

In this case, Theorem 4 reduces to the following classical result:

Theorem 5 (Báyer, Neukirch)**.** *Let X be a variety of dimension d over the finite field $\mathbb{F}$. Assume $\ell \neq p$. Let n be an integer such that the cohomology groups $\mathrm{H}^i_c(X,\mathbb{Z}_\ell(n)_X)$ are finite for all i. Then*

$$\left|Z(X,q^{-n})\right|_\ell = \prod_{i=0}^{2d+1} (\sharp\, \mathrm{H}^i_c(X,\mathbb{Z}_\ell(n)_X))^{(-1)^{i+1}}.$$