On symmetric transformations in metric measure geometry

Zusammenfassung

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The central objects of study in this thesis are metric measure spaces. These are metric spaces which are endowed with a reference measure and enriched with basic topological, geometric and measure theoretical properties. The objective of the first part of the work is to study the existence of a differential structure on symmetry groups of metric measure spaces. The second part is concerned with the analysis of the induced geometry of spaces on which there exists a symmetric action.

We study the group of measure-preserving isometries and the group of isometries of a metric measure space. We will consider a class of metric measure spaces in which metric tangent cones are well behaved and provide a characterization of spaces, within this class, whose automorphism groups are—possibly 0-dimensional—smooth manifolds, namely Lie groups. This is our first main result.

By using this conclusion we can further prove that automorphism groups are smooth in spaces with good optimal mass transport properties. In particular Riemannian manifolds, Alexandrov spaces of curvature bounded below, Ricci Limit spaces and their Finsler counterparts, enjoy these transport properties; this compliments well-known results. Furthermore, some spaces satisfying generalized Ricci curvature lower bounds enjoy these optimal transport properties as well. Curvature-dimension conditions define a notion of lower Ricci curvature bounds for metric measure spaces.

Theorem. *Well-behaved spaces that satisfy a curvature-dimension condition have smooth isomorphism groups.*

Examples of spaces which satisfy the hypothesis of the theorem are presented by finite dimensional RCD* spaces and, granted they have well-behaved tangents, their Finsler counterparts: strong CD spaces; strong CD* spaces; and essentially non-branching MCP spaces. However, we will illustrate that not all curvature-dimension conditions are sufficiently restrictive to guarantee smooth symmetry groups.

Next we turn our focus towards submetries, a metric analogue of Riemannian submersions. It is known that lower sectional curvature bounds are preserved—possibly in a synthetic manner—under such maps. However, examples have been constructed of Riemannian submersions that do not preserve lower bounds on the Ricci curvature tensor. On the other hand, by looking at a weighted, and eventually, a synthetic interpretation of Ricci curvature positive partial results in this direction have been achieved. The purpose of the second part of the thesis is to show the corresponding curvature stability results in full generality for curvature-dimension conditions.

Theorem. *Curvature-dimension conditions are preserved by bounded metric measure submetries.*

Metric measure submetries are particular submetries that respect the structure of the spaces that we study; they are in correspondence with a special type of foliations, called bounded metric measure foliations. This result implies that curvature-dimension conditions are stable under quotient maps induced by compact isomorphic group actions. As a byproduct of these results we obtain new constructions of examples of MCP, CD, CD^*, and RCD^* spaces.