Summary of the results of the doctoral thesis

ON THE $\overline{\partial}$-PROBLEM WITH POLYNOMIAL WEIGHTS AND LEVI-DEGENERATE HYPERSURFACES IN $\mathbb{CP}^n$

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This thesis is concerned with the question of existence and regularity of the solutions of nonhomogeneous Cauchy-Riemann equations

$$\overline{\partial}u(z) = f(z), \qquad z \in \Omega,$$

where Ω is a domain in a complex manifold. It turns out that in order to solve them one needs to impose some restrictions on the geometry of the domain Ω. For example, if Ω satisfies some *pseudoconvexity* property, this problem can be solved in the C^∞-category. Results on global regularity of the solutions produce some geometric applications and Siu's work on the nonexistence of smooth *Levi-flat* hypersurfaces in projective spaces gives one example of such an application. The information on the support behaviour can be used, for example, to solve the holomorphic extension problems for CR functions.

The $\overline{\partial}$-*problem with support condition* is the following question:
Given a domain Ω *in a complex manifold* X *and* $f \in C^\infty_{p,q}(X) \cap \operatorname{Ker} \overline{\partial}$ *with* $\operatorname{supp} f \subseteq \overline{\Omega}$*, can one find a smooth* $(p, q-1)$*-form* u *on* X *such that* $\overline{\partial}u = f$ *and* $\operatorname{supp} u \subset \overline{\Omega}$*?*

This problem was previously solved for *relatively compact* domains with regular boundaries, satisfying some pseudoconvexity property. In the first part of the thesis we investigate the $\overline{\partial}$-problem with support condition on *unbounded* domains in $\mathbb{C}^n$, without making any restriction on the smoothness of the boundary. However, it already cannot be solved without some global condition on f and our main purpose is to describe the class of forms and the properties of domains, for which it can be solved. We reach this aim by working with Sobolev spaces $H^k_{p,q}(\Omega, -|z|^2)$ with polynomial weights.

Our first result is as follows:
Let $\Omega \subsetneqq \mathbb{C}^n$ *be an open convex domain and* $f \in C^{\infty}_{p,q}(\mathbb{C}^n) \cap \operatorname{Ker} \overline{\partial}$, $0 \le p \le n$, *with* $\operatorname{supp} f \subset \overline{\Omega}$ *satisfies* $f\big|_{\Omega} \in \bigcap_{k\in\mathbb{N}} H^k_{p,q}(\Omega, -|z|^2)$. *If* $1 \le q \le n-1$, *then there exists* $u \in C^{\infty}_{p,q-1}(\mathbb{C}^n)$ *with* $\operatorname{supp} u \subseteq \overline{\Omega}$ *such that* $\overline{\partial} u = f$.

As an application we obtain:
Let $\Omega \subsetneqq \mathbb{C}^n$ *be convex and* $f \in C^{\infty}_{p,q}(\partial\Omega)$, $0 \le q \le n-2$ *be* $\overline{\partial}_b$*-closed on* $\partial\Omega$. *If* f *admits a smooth extension* $\tilde{f} \in \mathbb{C}^{\infty}_{p,q}(\mathbb{C}^n)$ *such that* $\overline{\partial}\tilde{f} \in \bigcap_{k\in\mathbb{N}} H^k_{p,q}(\Omega, -|z|^2)$ *and* $\overline{\partial}\tilde{f}$ *vanishes to infinite order at the boundary, then there exists a* $\overline{\partial}$*-closed extension of* f *to* Ω.

In particular, this result contains the known statement that every smooth compactly supported CR function on $\partial\Omega$ admits a holomorphic extension to Ω.

In the second part of the thesis, we investigate the question of nonexistence of smooth *Levi-degenerate* hypersurfaces in projective spaces. Several nonexistence results were obtained previously for *Levi-flat* CR manifolds. In this work we show that in general the answer to this question depends essentially on the signature of the hypersurface.

First, we prove the regularity for the tangential Cauchy-Riemann problem in bidegree $(0,1)$:
Let M *be a smooth real hypersurface in* $\mathbb{CP}^n$, $n \ge 2$, *which has a constant signature* (q^-, q^0, q^+) *with* $q^0 + \min\{q^-, q^+\} \ge 2$ *and* $q^0 \ge 1$ *at each point of* M. *Then for every* $k \in \mathbb{N}$ *and* $f \in C^{\infty}_{0,1}(M)$ *with* $\overline{\partial}_b f = 0$ *there exists* $u \in C^k(M)$ *satisfying the tangential Cauchy-Riemann equation* $\overline{\partial}_b u = f$ *on* M.

The above regularity together with Siu's arguments yields the main nonexistence result:
There exists no smooth real hypersurface in $\mathbb{CP}^n$, $n \ge 2$, *whose Levi form has constant rank and satisfies one of the following conditions:*

(*i*) *the Levi form has at least two zero eigenvalues,*

(*ii*) *the Levi form has at least one zero eigenvalue and two eigenvalues of opposite signs.*