On the spin cobordism invariance of the homotopy type of the space of metrics with invertible Dirac operator

Abstract

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The central topic of this thesis is the study of the homotopy type of the space of metrics with invertible Dirac operator on a connected closed spin manifold M.
The Dirac operator is a first order elliptic operator acting on sections of the spinor bundle of a spin manifold. The definitions of both the Dirac operator itself and the spinor bundle depend upon the metric of the underlying spin manifold M, it was shown by M. Atiyah and I. Singer that the index of the Dirac operator, defined as the difference between the dimension of its kernel and the dimension of its cokernel equals a topological quantity α depending only on the spin cobordism class of M.
Whenever $\alpha = 0$ (e.g. for spin nullcobordant manifolds) by a work of B. Ammann, M. Dahl and E. Humbert, the manifold M admits a metric for which the Dirac operator is invertible and the space of such metrics in dense in the set of all riemannian metrics on M.
We will provide several clues to validate the conjecture that the spaces of metrics with invertible Dirac operator on manifolds which are spin cobordant are weakly homotopy equivalent.
In particular we prove, by parametrizing the construction of B. Ammann, M. Dahl and E. Humbert, that the space of metrics with invertible Dirac operator is homotopy equivalent to the subspace of metrics $\mathcal{R}^{\mathrm{inv}}_{\frac{1}{2}\mathrm{flat}}(M)$ of half flat form near an embedded sphere S with trivial normal bundle. The space $\mathcal{R}^{\mathrm{inv}}_{\frac{1}{2}\mathrm{flat}}(M)$ is then proved to be weakly homotopy equivalent to the space $\mathcal{R}$, which is the union of the space $\mathcal{R}^{\mathrm{inv}}_{\mathrm{cyl}}(M \setminus S)$ of metrics on $M \setminus S$ which have cylindrical form near S and some of its accumulation points (metrics on M that are flat in the normal direction of S) with respect to a suitable topology. We prove further steps towards the last weakly homotopy equivalence, in particular we show some properties that suggest that the nullhomotopy needed to the weak homotopy equivalence $\mathcal{R}^{\mathrm{inv}}_{\mathrm{cyl}}(M \setminus S) \simeq \mathcal{R}$ is the one described in the thesis.
We conclude by suggesting how to use these results to the study of existence of metrics with nontrivial kernel of the Dirac operator on spin manifolds of dimension 4 and 5.