(Relaxed) Product Structures of Graphs and Hypergraphs

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Abstract

We investigate graphs and hypergraphs that have (relaxed) product structures.

In the class of graphs, we discuss in detail *RSP-relations*, a relaxation of relations fulfilling the square property and therefore of the product relation σ, that identifies the copies of the prime factors of a graph w.r.t. the Cartesian product. For $K_{2,3}$-free graphs finest RSP-relations can be computed in polynomial-time. In general, however, they are not unique and their number may even grow exponentially. Explicit constructions of such relations in complete and complete bipartite graphs are given.

Furthermore, we establish the close connection of (*well-behaved*) RSP-relations to (quasi-)covers of graphs and equitable partitions. Thereby, we characterize the existence of non-trivial RSP-relations by means of the existence of spanning subgraphs that yield quasi-covers of the graph under investigation. We show, how equitable partitions on the vertex set of a graph G arise in a natural way from well-behaved RSP-relations on $E(G)$. These partitions in turn give rise to quotient graphs that have rich product structure even if G itself is prime. This product structure of the quotient graph is still retained even for RSP-relations that are not well-behaved. Furthermore, we will see that a (finest) RSP-relation of a product graph can be obtained easily from (finest) RSP-relations on the prime factors w.r.t. certain products and in what manner the quotient graphs of the product w.r.t such an RSP-relation result from the quotient graphs of the factors and the respective product.

In addition, we examine relations on the edge sets of *hyper*graphs that satisfy the grid property, the hypergraph analog of the square property. We introduce the *strong* and the *relaxed* grid property as variations of the grid property, the latter generalizing the relaxed square property. We thereby show, that many, although not all results for graphs and the (relaxed) square property can be transferred to hypergraphs. Similar to the graph case, any equivalence relation R on the edge set of a hypergraph H that satisfies the relaxed grid property induces a partition of the vertex set of H which in turn determines quotient hypergraphs that have non-trivial product structures. Besides, we introduce the notion of *(Cartesian) hypergraph bundles*, the analog of (Cartesian) graph bundles and point out the connection between the grid property and hypergraph bundles.

Finally, we show that every connected thin hypergraph H has a unique prime factorization with respect to the normal and strong (hypergraph) product. Both products

coincide with the usual strong *graph* product whenever H is a graph. We introduce the notion of the Cartesian skeleton of hypergraphs as a natural generalization of the Cartesian skeleton of graphs and prove that it is uniquely defined for thin hypergraphs. Moreover, we show that the Cartesian skeleton of thin hypergraphs and its PFD w.r.t. the strong and the normal product can be computed in polynomial time.