LARGE DEVIATIONS FOR BROWNIAN INTERSECTION MEASURES

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Let us start with a number of independent Brownian motions from the origin in $\mathbb{R}^d$. Can one expect them to meet again in the future? We are interested in points in the space which are hit by all the motions, possibly at different time points. It turns out that arbitrarily many paths intersect in $\mathbb{R}^2$; while in $\mathbb{R}^3$, at most two paths intersect. Let S be this set of intersections. There is a natural measure which sits on S, and counts how intense the paths intersect each other. This measure, called the *Brownian intersection measure* can be symbolically given as:

$$\ell_t(A) = \int_A dy \quad \prod_{i=1}^p \int_0^{t_i} ds \quad \delta_y(W_s^{(i)}) \qquad \text{for every } A \subset \mathbb{R}^d \text{ Borel.} \tag{0.1}$$

Note that In $d \geq 2$, the above symbolical formula needs a rigorous justification (as the Brownian occupation measure fails to have a density). However, heuristically, the density of ℓ_t is the p-fold pointwise densities of each occupations measure $\ell_t^{(i)}$, which is defined as

$$\ell_t^{(i)}(A) = \int_0^t \mathrm{d}s\, \mathbb{1}_A(W_s^{(i)}).$$

This is one of the salient features of intersection measures: how far can these be understood as the product of the occupation measures? It is one of the goals of this thesis to answer this question in the large-t limit in the language of what is known as *large deviations.*

In the early 1970s, Donkser and Varadhan investigated the large-t behavior of the occupation time measures $\ell_t^{(i)}$. Although in $d \geq 2$, $\ell_t^{(i)}$ fails to have a density, it turns out that in the limit $t \uparrow \infty$ the densities appear: Let $\mathcal{M}_1(B)$ be the space of probability measures on a bounded open set B of $\mathbb{R}^d$ and τ_i be the exit time of the ith motion from B. Then, under the sub-probability density $\mathbb{P}_t(\cdot) = \mathbb{P}(\cdot \cap \{\tau_i > t\})$, $\frac{1}{t}\ell_t^{(i)}$ satisfies a large deviation principle in $\mathcal{M}_1(B)$, as $t \to \infty$ and rate function Λ. More explicitly, for $i = 1, \dots, p$ and $\mu^{(i)} \in \mathcal{M}_1(B)$,

$$\lim_{t\uparrow\infty} \frac{1}{t} \log \mathbb{P}_t\left(\frac{1}{t}\ell_t^{(i)} \approx \mu^{(i)}\right) = -\Lambda(\mu^{(i)})$$

where

$$\Lambda(\mu^{(i)}) = \begin{cases} \frac{1}{2}\big\|\nabla\psi_i\big\|_2^2 & \text{if } \psi_i^2 := \frac{\mathrm{d}\mu^{(i)}}{\mathrm{d}x} \in H_0^1(B) \\ \infty & \text{else.} \end{cases}$$

Therefore, ψ_i^2 manifests as the large-t density of the ith occupation masure. The heuristic formula (0.1) beacons that the pointwise product $\prod_{i=1}^p \psi_i^2$ should describe the large-t density of ℓ_t. The Theorem below, the main result of this thesis makes precise this statement.

Theorem 0.1 (LDP at diverging time)**.** *The tuple*

$$\Big(\frac{1}{t^p}\ell_t; \frac{1}{t}\ell_t^{(1)}, \dots, \frac{1}{t}\ell_t^{(p)}\Big)$$

satisfies, as $t \to \infty$, a large deviation principle in the space $\mathcal{M}(B) \times \mathcal{M}_1(B)^p$ under $\mathbb{P}_t$ with speed t and rate function

$$I\big(\mu; \mu_1, \dots, \mu_p\big) = \frac{1}{2}\sum_{i=1}^p \|\nabla\psi_i\|_2^2, \tag{0.2}$$

if $\mu, \mu_1, \dots, \mu_p$ each have densities ψ^{2p} and $\psi_1^2, \dots, \psi_p^2$ with $\|\psi_i\|_2 = 1$ for $i = 1, \dots, p$ such that $\psi, \psi_1, \dots, \psi_p \in H_0^1(B)$ and $\psi^{2p} = \prod_{i=1}^p \psi_i^2$; otherwise the rate function is ∞. The level sets of the rate function I in (0.2) are compact.

As we expected, Theorem 0.1 is an extension of the Donsker-Varadhan LDP for the occupation measures of a single Brownian motion to the intersection measure. It gives a rigorous meaning to the heuristic formula in (0.1) in the limit $t \to \infty$. Since B is bounded, ℓ_t is a finite measure. However, there is no natural normalisation of ℓ_t that turns it into a probability measure. Our result shows that $t^{-p}\ell_t$ is asymptotically of finite order.

Specialising to the first entry of the tuple, we get the following principle from the contraction principle:

Corollary 0.2. *The family of measures $(t^{-p}\ell_t)_{t>0}$ satisfies, as $t \to \infty$, a large deviation principle in the space $\mathcal{M}(B)$ under $\mathbb{P}_t$ with speed t and rate function*

$$I(\mu) = \inf\Big\{\frac{1}{2}\sum_{i=1}^p \|\nabla\psi_i\|_2^2\colon \psi_i \in H_0^1(B), \|\psi_i\|_2 = 1\, \forall i = 1, \dots, p, \text{ and } \prod_{i=1}^p \psi_i^2 = \frac{\mathrm{d}\mu}{\mathrm{d}x}\Big\}, \tag{0.3}$$

if μ has a density, and $I(\mu) = \infty$ otherwise. The level sets of the rate function I in (0.3) are compact.

A second version of our principle is proved for the motions observed until the individual exit times $\tau_1, \dots, \tau_p$ from B, i.e., the time horizon $(t_1, \dots, t_p) \in (0, \infty)^p$ gets replaced by $(\tau_1, \dots, \tau_p)$ and subsequently ℓ_t transforms to $\ell = \ell_\tau$. König/Mörters recently studied the upper tails of ℓ in a compact set $U \subset B$. They showed that under the conditional measure $\mathbb{P}(\cdot\,|\ell(U) > a)$, the intersection measure $\ell/\ell(U)$ satisfies a law of large numbers, as the intersection mass $a \uparrow \infty$. In Theorem 0.3 we characterise the exact logarithmic rate of the convergence. Let $\mathcal{M}_U(B)$ be the set of finite measures on B whose restriction to U is a probability measure.

Theorem 0.3 (Large deviations at diverging mass)**.** *The normalized probability measures $\ell/\ell(U)$ under $\mathbb{P}(\cdot \cap \{\ell(U) > a\})$ satisfy, as $a \to \infty$, a large deviation principle in the space $\mathcal{M}_U(B)$, with speed $a^{1/p}$ and rate function J, where*

$$J(\mu) = \inf\Big\{\frac{1}{2}\sum_{i=1}^p \|\nabla\phi_i\|_2^2\colon \phi_1, \dots, \phi_p \in H_0^1(B), \prod_{i=1}^p \phi_i^2 = \frac{\mathrm{d}\mu}{\mathrm{d}x}\Big\}, \tag{0.4}$$

if μ has a density and $J(\mu) = \infty$ otherwise.The level sets of J are compact.