A variational approach to travelling waves in particle chains with on-site potential

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This work is concerned with travelling waves in an infinite chain of particles where nearest neighbours interact and which is subjected to an external on-site potential. Mathematically, this chain is described by a system of infinitely many ordinary differential equations,

$$\ddot{u}_n(t) = V'\big(u_{n+1}(t) - u_n(t)\big) - V'\big(u_n(t) - u_{n-1}(t)\big) - K\sin(u_n(t)), \qquad K > 0, \tag{1}$$

for all $n \in \mathbb{Z}$; we look at several classes of interaction potentials V. *Travelling waves* with speed c are solutions of the form

$$u_n(t) = u(ct + n), \tag{2}$$

i.e., all atoms perform the same movements, just at different times. In other words, we seek solutions to the advance-delay differential equation

$$c^2 u''(z) = V'\big(u(z+1) - u(z)\big) - V'\big(u(z) - u(z-1)\big) - K\sin(u(z)). \tag{3}$$

Three different types of solutions to this equation are considered:

- heteroclinic travelling waves: $\lim_{z\to-\infty} u(z) = -\pi,$ and $\lim_{z\to+\infty} u(z) = \pi,$
- homoclinic travelling waves: $\lim_{z\to-\infty} u(z) = \lim_{z\to+\infty} u(z) = \pi,$
- periodic travelling waves: $u(z) = u(z+T)$ for some $T > 0$ and all $z \in \mathbb{R}$.

The asymptotic states are local maxima of the on-site potential energy.

It is shown that, for each of the three situations, there exists an open parameter regime for which such solutions exist. The solutions are constructed as critical points of functionals of the type

$$J(u) := \int_I \left[\frac{c^2}{2}\big[u'(z)\big]^2 - V\big(u(z+1) - u(z)\big) + K\big[1 + \cos(u(z))\big]\right] \mathrm{d}z, \tag{4}$$

where the domain of the functional, the domain of integration ($I = \mathbb{R}$ or $I = (0,T)$) in its definition, and the way how (3) can be interpreted as the Euler-Lagrange equation for J varies from chapter to chapter.

Proof techniques include—besides saddle point arguments—a new trichotomy result in the spirit of Lions' concentration compactness technique, along with a priori bounds obtained by comparison with Modica-Mortola type functionals and penalisation.