On curvature conditions using Wasserstein spaces

This thesis is twofold. In the first part, a proof of the interpolation inequality along geodesics in p-Wasserstein spaces is given and a new curvature condition on abstract metric measure spaces is defined.

In the second part of the thesis a proof of the identification of the q-heat equation with the gradient flow of the Renyi $(3-p)$-Renyi entropy functional in the p-Wasserstein space is given. For that, a further study of the q-heat flow is presented including a condition for its mass preservation.

Curvature condition. The proof of the Borel-Brascamp-Lieb inequality for Riemannian manifolds in [CEMS01] by Cordero-Erausquin, McCann and Schmuckenschläger, and later for Finsler manifolds by Ohta [Oht09], led Lott and Villani [LV09, LV07] and Sturm [Stu06a, Stu06b] to a new notion of a lower bound on the generalized Ricci curvature for metric measure spaces, called curvature dimension. In the first part of the thesis, p-Wasserstein spaces and the regularity of Kantorovich potential in the smooth setting are studied. Then adapting Ohta's proof [Oht09] the interpolation inequality along p-Wasserstein geodesic is shown.

This immediately leads to a new curvature condition using the ideas of Lott-Villani and Sturm. Under these conditions it is shown that a metric variant of Brenier's theorem can be proven and that a q-Laplacian comparison theorem holds for the subclass of spaces which are called q-infinitesimal convex spaces. In the last part, the theory of Orlicz-Wasserstein spaces is developed and necessary adjustments to prove the interpolation inequality along geodesics in those spaces are given and similarly one can define a curvature conditions using thoses spaces. However, due to the lack of a "vertical dual" and a well-defined Orlicz-Laplacian we are not able to get a similar Laplacian comparison theorem.

Heat and gradient flows. In [JKO98] Jordan, Kinderlehrer and Otto showed in the Euclidean setting that one can identify the heat flow with the gradient flow of the entropy functional in the 2-Wasserstein space. Later Ambrosio, Gigli and Savaré [AGS13] gave a proof of the identification on abstract metric measure spaces by showing that an abstractly defined heat equation solves the gradient flow problem of the entropy functional in the 2-Wasserstein space. In this part of the thesis, a calculus along the q-heat flow, i.e. the gradient flow of the q-Cheeger energy $f \mapsto \frac{1}{q}\int |\nabla f|^q d\mu$, is developed. After showing mass preservation under a generalized growth condition on the measure, expressed as $\int V^p \exp_p(-V^p) d\mu < \infty$, where $V(x) = C(1 + d(x, x_0))$, it is shown that the q-heat flow solves a generalized gradient flow problem of the $(3-p)$-Renyi entropy functional

$$E : f \mapsto \frac{1}{(3-p)(2-p)} \int f^{3-p} - f d\mu$$

in the p-Wasserstein space. This requires lower semicontinuity of the descending slop $|D^- E|$ which is implied by the curvature condition defined in this thesis. In case $q \geq 2$ one can identify the two flow by using a convexity property of the derivative of E along the q-heat flow. This derivative is called the q-Fisher information $f_t \mapsto -\int \frac{|\nabla f_t|^q}{f^{p-1}} d\mu$ and agrees with the well-known Fisher information in case $q = p = 2$.

References

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