Summary of the Dissertation

Floer homology for homoclinic tangles

by

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This work points out a relation between two important topics of symplectic dynamical systems — homoclinic points and Lagrangian Floer homology. Based on this we construct a new symplectic invariant for homoclinic tangles:

Primary homoclinic Floer homology

Let (M, ω) be $(\mathbb{R}^2, dx \wedge dy)$ or a symplectic closed two-dimensional manifold with genus $g \geq 1$ and let φ be a symplectomorphism with hyperbolic fixed point x. For symplectomorphisms the (un)stable manifolds $L_0 := W^u(x, \varphi)$ and $L_1 := W^s(x, \varphi)$ are *Lagrangian* submanifolds. Thus the set of *homoclinic* points $\mathcal{H} := L_0 \pitchfork L_1$ can be seen as the intersection set associated to the *noncompact Lagrangian intersection problem* (L_0, L_1). This motivates the construction of (Lagrangian) Floer homology for homoclinic tangles.

There is a $\mathbb{Z}$-action on $\mathcal{H}$. For transversely intersecting $L_0 \pitchfork L_1$ the set $\mathcal{H}/\mathbb{Z}$ is still infinite. This prevents the well-definedness of the usual Floer differential on $\mathcal{H}$. Moreover the action filtration admits neither finite sup- nor finite sublevel sets (mod $\mathbb{Z}$).

Nevertheless there is a natural subset of $\mathcal{H}$ on which the Floer differential is well-defined. Denote by $[p, q]_i$ the segment between p and q in L_i for $i \in \{0, 1\}$. We call p contractible if the loop $[p, x]_0 \cup [p, x]_1$ is contractible and denote by $\mathcal{H}_{[x]} \subset \mathcal{H}$ the set of contractible homoclinic points. Then

$$\mathcal{H}_{pr} := \{p \in \mathcal{H}_{[x]} \backslash \{x\} \mid]p, x[_0 \cap \,]p, x[_1 \cap \mathcal{H}_{[x]} = \emptyset\}$$

is the set of *primary* homoclinic points. $\tilde{\mathcal{H}}_{pr} := \mathcal{H}/\mathbb{Z}$ is finite and we denote the equivalence class of $p \in \mathcal{H}_{pr}$ by $\langle p \rangle$. The Maslov index μ induces a grading on

$\tilde{\mathcal{H}}_{pr}$ and we define, analogously to classical Lagrangian Floer homology,

$$C_k := C_k(x,\varphi) := \bigoplus_{\substack{\langle p\rangle \in \tilde{\mathcal{H}}_{pr} \\ \mu(\langle p\rangle)=k}} \mathbb{Z}\langle p\rangle,$$

$$\partial\langle p\rangle := \sum_{\substack{\langle q\rangle \in \tilde{\mathcal{H}}_{pr} \\ \mu(\langle q\rangle)=\mu(\langle p\rangle)-1}} m(\langle p\rangle,\langle q\rangle)\langle q\rangle,$$

$$H_* := H_*(x,\varphi) := \frac{\ker\partial}{\operatorname{Im}\partial}.$$

The well-definedness of ∂ and the proof of $\partial\circ\partial = 0$ are tricky combinations of dynamical and combinatorial arguments.

H_* is invariant under so called *contractibly strongly intersecting (symplectic) isotopies.* The proof has to combine analytical and combinatorial arguments. Note that a primary point $p \in \mathcal{H}_{pr}$ might vanish (analogously arise) in two ways:

- p vanishes as intersection point,
- p persists as intersection point, but is no longer primary.

The invariance implies an existence and bifurcation criterion for homoclinic points and the fixed point. In the two-dimensional situation H_* also can be defined for nonsymplectic diffeomorphisms, but there is no natural invariance. Thus H_* is an symplectic invariant.

H_* is invariant under conjugacy. Moreover we compare $H_*(x,\varphi)$ and $H_*(x,\varphi^n)$. *Chaotic primary homoclinic Floer homology* takes also the chaos near a homoclinic tangle into account and gives rise to a symplectic zeta function. Furthermore we define the action spectrum and action filtration of primary homoclinic Floer homology and investigate their properties.
Then we analyse the problems which prevent differential graded algebras or $\mathcal{A}_\infty$-structures based on (primary) homoclinic points.

Finally we sketch a stronger invariance theorem and applications to Birkhoff invariants. Moreover we briefly discuss the problems arising on higher dimensional manifolds.

H_* is the first invariant which takes the *algebraic* interaction of homoclinic points into account. Moreover H_* simultanously is a *semi-global and semi-local* invariant: On the one hand the branches and homoclinic points can lie anywhere on the manifold, but on the other hand we are bound to contractible points. Thus the topology of the manifold enters only indirectly: If $H_*(x,\varphi) = 0$ then either L_0 and L_1 do not intersect or there are no *contractible* homoclinic points. There is no direct way to relate $H_*(x,\varphi)$ to the topology of M or L_0 and L_1.