Characterizations of Recognizable Picture Series

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Abstract

The theory of two-dimensional languages as a generalization of formal string languages was motivated by problems arising from image processing and pattern recognition and also concerns models of parallel computing. In the nineties, Restivo and Giammarresi defined the family REC of *recognizable picture languages* [GR97]. This family is very robust and has been characterized by many different devices, generalizing well-known equivalences of regular word languages. Several authors obtained an equivalence theorem for picture languages describing recognizable languages in terms of types of automata, sets of tiles, rational operations or existential monadic second-order (MSO) logic [BG05, GRST96, IN77, LS97].

Here we investigate *picture series*. These are functions that map pictures, i.e. arrays of symbols, over a finite alphabet to elements of a commutative semiring and provide an extension of two-dimensional languages to a quantitative setting.

The model of a weighted (quadrapolic) picture automaton (WPA) was introduced by Bozapalidis and Grammatikopoulou [BG05]. WPA are automata operating in a natural way on pictures and whose transitions carry weights. The behavior or the computation of a WPA is a picture series. The family of picture series over the alphabet Σ and a commutative semiring K that are computable by WPA will be denoted by $K^{\mathrm{rec}}\langle\langle\Sigma^{++}, WPA\rangle\rangle$.

Bozapalidis and Grammatikopoulou showed that picture series computed by WPA are closed under certain operations and projections on series. These operations on finite languages define *rational* picture series. The class of series that are projections of rational series are collected in $K^{Prat}\langle\langle\Sigma^{++}\rangle\rangle$.

We will prove that, for commutative semirings, the behaviors of weighted picture automata are precisely alphabetic projections of series defined in terms of rational operations and also coincide with the families of series characterized by weighted tiling or weighted domino systems. We thus obtain a robust definition of a class of recognizable picture series. We get the results for languages by restricting the semiring to the Boolean semiring.

Theorem 1. *Let K be a commutative semiring and let Σ be an alphabet. Then*

$$K^{rec}\langle\langle\Sigma^{++}, WPA\rangle\rangle = K^{Prat}\langle\langle\Sigma^{++}\rangle\rangle.$$

In the course of the thesis we will investigate new properties of unambiguous picture languages. We will apply the notion of unambiguity to several devices describing picture languages and prove an equivalence theorem similar to the known basic equivalences in the theory of two-dimensional languages.

One cause which motivated investigating unambiguity in the context of picture languages also were properties needed in the further course of this thesis for the closure of logic-definable series under universal first-order quantification.

Proposition 1. *Let Σ be any alphabet and let φ be a first-order formula on Σ. Then φ defines a picture language that is unambiguous.*

Recently, Droste and Gastin ([DG05] introduced the framework of a weighted logic over words and characterized recognizable formal power series, computed by weighted finite automata, as semantics of monadic second-order sentences within their logic.

One part of this dissertation is devoted to the concept of a weighted MSO logics for pictures. The semantics of a weighted formula will be a picture series.

For the syntax, we basically follow classical logic. But additionally, similar to [DG05], we also let elements of the semiring be atomic formulas, hence again we are able to formulate quantitative properties of picture languages: imagine for instance the number of a's occurring in a picture. We apply negation only to atomic formulas. Universal quantification in general does not preserve recognizability. Hence, as in [DG05], we disallow universal set quantification, but here we restrict universal first-order (FO) quantification in a new way to particular formulas (defining FO step functions).

Our main result here proves that for an alphabet and any commutative semiring the family of picture series computable by WPA coincides with the family of series that are definable by weighted monadic second-order sentences that are in existential form and which are in a certain sense restricted. The family of series that are the semantics of weighted MSO sentences in restricted and existential form will be denoted by $K^{\mathrm{remso}}\langle\langle\Sigma^{++}\rangle\rangle$.

We introduce weighted 2-dimensional on-line tessellation automata (W2OTA) computing picture series and extending the common automata-theoretic model for picture languages [IN77]. We denote the class of series computed by W2OTA by $K^{\mathrm{rec}}\langle\langle\Sigma^{++}, W2OTA\rangle\rangle$.

Theorem 2. *Let Σ be an alphabet and K any commutative semiring. Then*

$$K^{\mathrm{rec}}\langle\langle\Sigma^{++}, WPA\rangle\rangle = K^{\mathrm{rec}}\langle\langle\Sigma^{++}, W2OTA\rangle\rangle = K^{\mathrm{remso}}\langle\langle\Sigma^{++}\rangle\rangle.$$

One immediate question now clearly is whether there exist semirings where we actually do not need the restriction, i.e. where the full EMSO logic corresponds to recognizable picture series. We will see that many results of the 1-dimensional case of word series are not longer true in our setting. Exemplary, for any commutative semiring there are recognizable series that are not FO step functions. For $\mathbb{Z}/2\mathbb{Z}$ there even exists a weighted FO formula whose semantics is not a FO step function. Nevertheless we show that in case the semiring is weakly bi-aperiodic then all EMSO-formulas define recognizable picture series.

Viewing at decidability problems we show that fundamental properties of weighted formulas and behaviors of weighted picture automata become undecidable for two dimensions.

Theorem 3. *Let K be any commutative semiring. It is undecidable whether a given weighted MSO formula over K is restricted. Furthermore, it is undecidable whether two given W2OTA compute identical picture series.*

Even to decide for certain semirings, whether a W2OTA computes a series with finite image differs from the word case and becomes undecidable if the underlying semiring is not locally finite.

We will amplify on properties of the supports of recognizable picture series. We show that in case the underlying semiring is commutative but not a ring, then every support of a recognizable characteristic series is recognizable. If the semiring is positive then every support of a recognizable series is again recognizable.

But we will also realize that due to the fact that recognizable picture languages are not closed under complement and that not every picture automaton can be determinized, only very rarely we get positive answers in the full range as they exist for the one dimensional case and there are as well negative results.

We will also elaborate on the special case where the underlying alphabet is assumed to be a singleton. In [dPV97] the authors consider the corresponding unary picture languages. In the weighted case, picture series can be represented by series over the free Abelian monoid $M = \{a\}^* \times \{b\}^*$ and the natural question arises, asking how rational series on M and recognizable picture series are related. We prove that for an arbitrary commutative semiring the family of rational series over M, interpreted as picture series on a unary alphabet, is strictly included in the family of series that are the behaviors of weighted picture automata.

A natural problem that arises is to ask for a general concept of weighted acceptance, as well as, for some weighted logics which includes existing approaches of structures that have been investigated before. Some of these objects can be seen as directed graphs of some particular structure. In [Tho91], Thomas introduces graph acceptors to define sets of words, trees and graphs. We will generalize a proper subclass of these graph acceptors to some weighted device containing the weighted models existing for pictures and trees. These weighted acceptors compute functions from certain directed graphs into some commutative semiring. Furthermore, we will determine some weighted MSO logics over graphs.

Proposition 2. *Every behavior of a weighted graph acceptor is the semantic of a sentence of the weighted MSO logics on graphs that is in restricted and existential form.*

If we restrict ourselves to the class graphs containing all pictures over an alphabet then the concept of weighted recognizability on graphs does coincide with the notion of a recognizable picture series in this thesis. Analogously, restricting the set of graphs to the class of bounded unranked trees we get the known family of recognizable tree series.

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