SUMMARY OF RESULTS:

THE COMPLETION OF THE MANIFOLD OF RIEMANNIAN METRICS WITH RESPECT TO ITS L^2 METRIC

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Let M be a smooth, closed, finite-dimensional, oriented manifold, and denote by $\mathcal{S}$ the Fréchet space of smooth, symmetric $(0,2)$-tensor fields on M. Let $\mathcal{M} \subset \mathcal{S}$ be the set of all Riemannian metrics on M. It is an open set of $\mathcal{S}$ and is therefore trivially a Fréchet manifold. Also, the tangent space at each point of $\mathcal{M}$ is canonically identified with $\mathcal{S}$.

There is a natural Riemannian metric on $\mathcal{M}$ called the L^2 metric and denoted by $(\cdot,\cdot)$. At the point $g \in \mathcal{M}$, it is given by

$$(h,k)_g := \int_M g^{ij} h_{il} g^{lm} k_{jm} \, d\mu_g \quad \text{for all } h,k \in \mathcal{S} \cong T_g\mathcal{M},$$

where μ_g is the volume form induced by g.

The Riemannian manifold $(\mathcal{M}, (\cdot,\cdot))$ has been studied in applications ranging from mathematical physics—in general relativity, for example—to the theory of Teichmüller and moduli spaces for Riemann surfaces. In my Ph.D. thesis, I proved several results on the metric geometry of $(\mathcal{M}, (\cdot,\cdot))$, three of which are stated here.

The first is the following:

Theorem 1. *With the Riemannian distance function d induced from $(\cdot,\cdot)$, $(\mathcal{M}, d)$ is a metric space.*

This is indeed a theorem that needs proving, as the L^2 metric on $\mathcal{M}$ is an example of a so-called weak Riemannian metric. This means that the tangent spaces of $\mathcal{M}$ are not complete with respect to $(\cdot,\cdot)$, and so many general theorems from the usual theory of Riemannian Hilbert manifolds do not hold. In this theory, typically only so-called strong metrics are considered, with respect to which the tangent spaces are complete. Of course, for a strong Riemannian Hilbert manifold, the Riemannian metric induces a metric space structure on the manifold. However, for weak Riemannian manifolds, there are examples where this does not hold, as the distance between some points may be zero. Therefore, one must explicitly prove that a given weak Riemannian manifold is a metric space.

The second main result is a description of the completion $\overline{\mathcal{M}}$ of $\mathcal{M}$ with respect to $(\cdot,\cdot)$. Let $\mathcal{M}_f$ denote the set of measurable semimetrics—symmetric $(0,2)$-tensor fields which induce a positive-semidefinite scalar product on each tangent space—on M with finite volume. Define an equivalence relation on $\mathcal{M}_f$ by saying $g_0 \sim g_1$ if the following statement holds for almost every $x \in M$: if g_0 and g_1 differ, then both g_0 and g_1 fail to be positive definite. Given this, we have:

Theorem 2. *There is a natural bijection $\Omega : \overline{\mathcal{M}} \to \mathcal{M}_f/\sim$ that is the identity when restricted to $\mathcal{M} \subset \overline{\mathcal{M}}$.*

The final result we describe here is an application of the completion of $\mathcal{M}$ to Teichmüller theory. If the base manifold M is additionally assumed to be a Riemann surface of genus larger than one, then the Teichmüller space $\mathcal{T}$ of M can be identified with the space of conformal classes of metrics on M modulo $\mathcal{D}_0$, by which we denote the diffeomorphisms of M that are homotopic to the identity. Let $\mathcal{N}$ be a smooth submanifold of $\mathcal{M}$ which is invariant under the action (by pull-back) of the diffeomorphism group, and which contains exactly one representative from each conformal class. Then we have a diffeomorphism $\mathcal{T} \cong \mathcal{N}/\mathcal{D}_0$, and the L^2 metric restricted to $\mathcal{N}$ induces a Riemannian metric on $\mathcal{T}$. As a corollary of the last theorem, we have:

Theorem 3. *Each point in the completion of $\mathcal{T}$ with respect to the Riemannian metric described above can be identified with an element of $\mathcal{M}_f/\sim$. This identification unique up to a choice of representative within a $\mathcal{D}_0$-equivalence class.*

The metrics on $\mathcal{T}$ just constructed generalize the Weil-Petersson metric on Teichmüller space, and this theorem generalizes what is already known about the completion of Teichmüller space with respect to the Weil-Petersson metric. In particular, if $\mathcal{N} = \mathcal{M}_{-1}$, the space of hyperbolic metrics on M (those with constant scalar curvature -1), then the metric obtained from the above construction coincides with the Weil-Petersson metric on Teichmüller space. It is known that the completion of this metric is obtained by adding in certain hyperbolic surfaces with cusps, which are in particular open hyperbolic manifolds with finite volume.