Eigenvalues of directed and undirected graphs and their applications

Frank Bauer

In this thesis, we studied eigenvalues of directed and undirected graphs and their applications. In the first part, a detailed study of the largest eigenvalue of the normalized Laplace operator Δ for undirected graphs was presented. In contrast to the smallest nontrivial eigenvalue λ_1, the largest eigenvalue λ_{n-1} has not been studied systematically before. However, it is well-known that λ_1 can be controlled from above and below in terms of the Cheeger constant h in the following way:

$$1 - \sqrt{1-h^2} \leq \lambda_1 \leq 2h \tag{1}$$

We showed that the largest eigenvalue satisfies the following dual version of the Cheeger estimate:

$$2\bar{h} \leq \lambda_{n-1} \leq 1 + \sqrt{1-(1-\bar{h})^2},$$

where $\bar{h}$ is a measure for how close the graph is to a bipartite one. In addition to the dual Cheeger estimate, we derived two complementary estimates for the largest eigenvalue. We introduced so-called neighborhood graphs in order to control the largest eigenvalue λ_{n-1}. In fact, the concept of neighborhood graphs is much more general and can also be used to obtain other eigenvalue estimates. For instance, by using the neighborhood graphs we were also able to derive new estimates for λ_1. These new estimates improve the Cheeger estimate (1) for some graphs. As a last estimate, we showed that the largest eigenvalue λ_{n-1} can be controlled by a local clustering coefficient.

As applications of our estimates for the largest eigenvalue λ_{n-1}, we studied the convergence of a random walk on a graph as well as the synchronizability of coupled map lattices. For random walks on graphs, the convergence to equilibrium is determined by the quantity $\max_{i\neq 0}|1-\lambda_i|$ and the synchronizability is determined by $\frac{\lambda_{n-1}}{\lambda_1}$. Using our results, we were able to give a rather complete picture of the influence of the graph structure on the convergence of a random walk (on the synchronizability of a coupled map lattice).

In the second part of this thesis, we introduced the normalized Laplace operator for directed graphs with positive and negative weights. In contrast to the normalized Laplace

operator for undirected graphs with nonnegative weights, this normalized Laplace operator is neither self-adjoint nor nonnegative. We studied the eigenvalues of the Laplace operator systematically and generalized many results for the Laplace operator for undirected graphs with nonnegative weights to the more general setting of directed graphs with positive and negative weights. In particular, we showed that the important class of directed acyclic graphs with nonnegative weights can be completely characterized by the spectrum of Δ. We proved different comparison theorems that established a relationship between the eigenvalues of directed graphs and the eigenvalues of certain undirected graphs. These comparison theorems were then exploited to control the real parts of the eigenvalues of a directed graph in a suitable way. We generalized the concept of neighborhood graphs to directed graphs with positive and negative weights. As in the case of undirected graphs, this concept is used to derive several new eigenvalue estimates.

In the third part of this thesis, we studied different important questions that arise in the study of complex networks. We identified an important mechanism that explains how complex, even chaotic, behavior can emerge in a network of dynamically simple units. Furthermore, we studied in detail how the network topology influence the synchronization properties of a complex network. Using our results, we could show that all balanced directed networks have better, or equally good, synchronization properties than the corresponding undirected network. This is interesting, since compared to directed networks, in undirected networks information always flows along both directions of the edge. Hence, our results shows that the intuitive assumption that better synchronizability is related to factors easing information flow in networks is in general not true and totally fails for balanced directed networks.