Summary

This work aims to investigate a special moment problem of Stieltjes-type:

$\mathsf{M}[[\alpha,\infty);(s_j)_{j=0}^m,\leq]$: Let $\alpha\in\mathbb{R}$, let $m\in\mathbb{N}_0$, and let $(s_j)_{j=0}^m$ be a sequence of complex $q\times q$ matrices. Parametrize the set $\mathcal{M}_{\geq}^q[[\alpha,\infty);(s_j)_{j=0}^m,\leq]$ of all non-negative Hermitian $q\times q$ measures σ for which $\int_{[\alpha,\infty)} t^j\sigma(dt)$ exists for every $0\leq j\leq m$. Furthermore, we demand that the matrix $s_m-\int_{[\alpha,\infty)} t^m\sigma(dt)$ is non-negative Hermitian and, in case $m\geq 1$, moreover, that $\int_{[\alpha,\infty)} t^j\sigma(dt)=s_j$ is fulfilled for every $0\leq j\leq m-1$.

After formulating some well-known necessary and sufficient conditions of solvability in Section 1, in Section 2 we present some results on certain classes of holomorphic matrix-valued functions and special integral transformations of non-negative measures. This leads us to an equivalent interpolation problem. The description of this problem by two different approaches is the main goal of this work and is presented in the Sections 3 and 4.

Hence, in Section 3 we deal with systems of Potapov's fundamental matrix inequalities and extend known results for a given sequence $(s_j)_{j=0}^{2n+1}$ with $n\in\mathbb{N}_0$ to the case of a given sequence $(s_j)_{j=0}^{2n}$ with $n\in\mathbb{N}_0$. Therefore, we examine these special systems of fundamental matrix inequalities and certain invariant subspaces, so-called Dubovoj subspaces. Moreover, we are interested in special matrix polynomials and the associated $\tilde{J}_q$ forms, where $\tilde{J}_q$ denotes the relevant signature matrix for the system. As part of our parameterization, we resort to certain pairs of meromorphic matrix-valued functions, so-called Stieltjes pairs, as free parameters.

In Section 4, we consider a Schur-analytic approach. Here, our goal is to obtain a parameterization of the solution set of the interpolation problem based on a data set $(s_j)_{j=0}^m$ with $m\in\mathbb{N}_0$. The Schur-type algorithm presented here consists of an algebraic and a function-theoretical part. The algebraic part includes those finite sequences that admit a solution to the moment problem. Based on the concept of the reciprocal sequence, these are later gradually shortened by one element in each step, preserving special properties of a certain sequence. Within the function-theoretical part all the holomorphic matrix-valued functions essential for the moment problem are included. Considerations on linear fractional transformations of matrices serve as an important tool for the development of the algorithm.

Both representations aim at a description of the solution in the non-degenerate case as well as in the different degenerate cases.