Unbounded Induced ∗-Representations.

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Induced modules are a fundamental tool in representation theory of groups and algebras. If $B$ is a subring of a ring $A$ and $V$ is a left $B$-module, then the left $A$-module $A \otimes_B V$ with action defined by \( a_0(a \otimes v) := a_0a \otimes v \) is called induced module of $V$.

Let $B \subset A$ be associative algebras over $\mathbb{C}$ with involution $*$ and let $V$ be a Hermitian $B$-module. That is, we have $\langle bx, y \rangle = \langle x, b^*y \rangle$ for all $x, y \in V$ and $b \in B$. In the first part of the thesis we investigate the problem of defining a corresponding induced Hermitian module. In order to do so, we have to introduce an appropriate inner product on the space $A \otimes_B V$ or on some quotient space.

This is done by means of conditional expectations. Let $A$ be a unital $*$-algebra and let $B$ be a unital $*$-subalgebra of $A$. A linear mapping $p : A \to B$ is called a conditional expectation from $A$ onto $B$ if the following conditions are satisfied:

1. $p(a^*) = (p(a))^*$, $p(b_1ab_2) = b_1p(a)b_2$ for all $a \in A$, $b_1, b_2 \in B$, $p(1_A) = 1_B$.
2. $p(\sum A^2) \subseteq B \cap \sum A^2$,

where $\sum A^2$ denotes the cone of all finite sums of elements $a^*a$, $a \in A$.

If there exists a conditional expectation $p$ from a $*$-algebra $A$ onto a $*$-subalgebra $B$ and if a unitary space $(V, \langle \cdot, \cdot \rangle)$ is a hermitian $B$-module, then there exists a sesquilinear form $\langle \cdot, \cdot \rangle_0$ on $A \otimes_B V$ defined by

\[
\langle a_1 \otimes v_1, a_2 \otimes v_2 \rangle_0 := \langle p(a^*_2a_1)v_1, v_2 \rangle.
\]

(1)

The module $V$ is called inducible if the form (1) is positive semidefinite. In this case the quotient space of $A \otimes_B V$ by the null space of the form $\langle \cdot, \cdot \rangle_0$ is a Hermitian $A$-module $D$. It extends uniquely to a $*$-representation of $A$ on the Hilbert space completion of $D$.

A standard method for the construction of conditional expectations of $C^*$-algebra is based on groups of $*$-automorphisms. We develop an analogue of this method for general $*$-algebras. Let $G$ be a compact group acting by $*$-automorphisms $\alpha_g$, $g \in G$, on a $*$-algebra $A$ and let $\mu$ be the normalized Haar measure of $G$. We say that the action $\alpha_g$ is locally finite-dimensional if for every $a \in A$ the linear hull of the set \( \{\alpha_g(a), \ g \in G\} \) is finite-dimensional. Then the map $a \mapsto \int \alpha_g(a)d\mu$, $a \in A$, is a well-defined conditional expectation from $A$ onto the $*$-subalgebra $B$ of $G$-stable elements.

An important class of algebras where conditional expectations arise in a natural manner are group graded $*$-algebras. Let $G$ be a discrete group. We say that a $*$-algebra $A$ is a $G$-graded if $A$ is a direct sum of subspaces $A_g$, $g \in G$, such that

\[
A_g \cdot A_h \subseteq A_{g+h} \quad \text{and} \quad (A_g)^* \subseteq A_{g^{-1}} \quad \text{for all} \ g, h \in G.
\]

For any subgroup $H \subseteq G$ the canonical projection $p_H$ from $A$ onto $A_H = \oplus_{g \in H} A_g$ is a conditional expectation. In this context we develop a theory of induced representations. Among others we prove various versions of the Imprimitivity Theorem.

In this thesis we are dealing with unbounded $*$-representations of $*$-algebras on Hilbert space. There is a striking difference to the theory of bounded representations: the problem of classifying all irreducible unbounded $*$-representations of a general $*$-algebra is not well-posed.
The second part of the thesis deals with “well-behaved” \(*\)-representations. The context in which we define well-behaved representations is the following. We take a \(G\)-graded unital \(*\)-algebra \(A = \bigoplus_{g \in G} A_g\). Further we assume that the \(*\)-subalgebra \(B := A_e\) is commutative. We denote by \(\hat{B}^+\) the space of all characters on \(B\) which are nonnegative on the cone \(\sum A^2 \cap B\). We will also assume that all characters \(\chi \in \hat{B}^+\) satisfy the following condition:

\[
\chi(c^*d)\chi(d^*c) = \chi(c^*c)\chi(d^*d)
\]

for all \(\chi \in \hat{B}^+, \ g \in G, \) and \(c, d \in A_g\).

Then we define a partial action of the group \(G\) on the set \(\hat{B}^+\). Let \(\chi \in \hat{B}^+, g \in G\). We say that \(\chi^g\) is defined if there exists an element \(a_g \in A_g\) such that \(\chi(a_g^*a_g) > 0\). In this case we put

\[
\chi^g(b) := \frac{\chi(a_g^*ba_g)}{\chi(a_g^*a_g)}, \ b \in B.
\]

Using this partial action we define a notion of well-behaved representations of \(A\). An essential part of the thesis is devoted to studying of fundamental properties of these well-behaved representations. Some of the properties are collected in the following

**Theorem.** Let \(\pi\) be a \(*\)-representation of \(A\), let \(H\) be a subgroup of \(G\) and let \(\rho\) be a \(*\)-representation of \(A_H\). Then the following statements hold.

(i) If \(\pi\) is bounded, then \(\pi\) is well-behaved.

(ii) If \(\pi\) is well-behaved, then \(\pi\) is self-adjoint and any self-adjoint sub-representation \(\pi_0 \subseteq \pi\) is well-behaved. In particular, every well-behaved sub-representation of \(\pi\) has a well-behaved complement.

(iii) If \(\pi\) is a well-behaved representation with metrizable graph topology, then \(\pi\) can be decomposed into a direct orthogonal sum of cyclic well-behaved representations.

(iv) If \(\rho\) is a well-behaved representation with metrizable graph topology, then \(\rho\) is inducible via \(p_H\) if and only if \(\rho\) is \((\sum A^2 \cap A_H)\)-positive.

(v) If \(\rho\) is a well-behaved inducible representation with metrizable graph topology, then the induced representation \(\text{Ind}_{A_H \uparrow A}(\rho)\) is well-behaved.

In this context we develop an analogue of the Mackey normal subgroup analysis. First we associate irreducible well-behaved representations to orbits under the partial action of \(G\) on \(\hat{B}^+\). A central result of our Mackey analysis is the following

**Theorem.** Let \(\chi \in \hat{B}^+\) and let \(H \subseteq G\) be its stabilizing subgroup. Then every irreducible well-behaved representation \(\pi\) associated to \(\text{Orb}_\chi\) is induced from a bounded irreducible \(*\)-representation \(\rho\) of the algebra \(A_H\) satisfying the following condition:

(2) \(\text{Res}_{\hat{B}^+} \rho\) corresponds to a multiple of the character \(\chi\).

Moreover, the representation \(\rho\) is uniquely by \(\pi\) and \(\chi\) up to unitary equivalence.

It is shown that large classes of important examples fit into the latter context. Among them are Weyl algebras, enveloping algebras of \(su(2)\) and \(su(1,1)\), quantized enveloping algebras \(U_q(su(2))\) and \(U_q(su(1,1))\), quotients of the enveloping algebra of the Virasoro algebra, \(*\)-algebras associated with dynamical systems, quantum disc algebras, Podles’ quantum spheres, quantum algebras, and others.